About one multidimensional sum with Fibonacci Numbers.

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This note motivated by the following problem: [1]
Let (f_n) be the Fibonacci sequence $f_0 = 0, f_1 = 1, f_{n+1} = f_n + f_{n-1}, n \in \mathbb{N}$.

Determine $h_n = \sum_{i,j,k} f_i f_j f_k$, where the sum is over $i, j, k > 0$ and $i + j + k = n$.

We will consider general problem of calculation sum

$$S_m(n) := \sum_{i_1,i_2,\dots,i_m \geq 0}^{i_1+i_2+\dots+i_m=n} f_{i_1} f_{i_2} \dots f_{i_m}$$

for any nonnegative integer m and $n.(S_0(n) = 0$ as sum by empty set).
Easy to see that in particular $S_m(0) = 0$ and $S_1(n) = f_n$.

We have $S_m(n) = \sum_{i_1,i_2,\dots,i_m, \geq 0}^{i_1+i_2+\dots+i_m=n} f_{i_1} f_{i_2} \dots f_{i_m} =$

$$\sum_{i_1,i_2,\dots,i_m, \geq 0}^{i_1+i_2+\dots+i_{m-1} \leq n} f_{i_1} f_{i_2} \dots f_{i_{m-1}} f_{n-(i_1+i_2+\dots+i_{m-1})} = \sum_{k=1}^{n-1} \sum_{i_1,i_2,\dots,i_m, \geq 0}^{i_1+i_2+\dots+i_{m-1}=k} f_{i_1} f_{i_2} \dots f_{i_{m-1}} f_{n-k}.$$

$$S_m(n+1) = \sum_{k=1}^{n} \sum_{i_1,i_2,\dots,i_m, \geq 0}^{i_1+i_2+\dots+i_{m-1}=k} f_{i_1} f_{i_2} \dots f_{i_{m-1}} f_{n+1-k} =$$

$$\sum_{k=1}^{n} \sum_{i_1,i_2,\dots,i_m, \geq 0}^{i_1+i_2+\dots+i_{m-1}=k} f_{i_1} f_{i_2} \dots f_{i_{m-1}} f_{n-k} + \sum_{k=1}^{n} \sum_{i_1,i_2,\dots,i_m, \geq 0}^{i_1+i_2+\dots+i_{m-1}=k} f_{i_1} f_{i_2} \dots f_{i_{m-1}} f_{n-1-k} =$$

$$\sum_{k=1}^{n-1} \sum_{i_1,i_2,\dots,i_m, \geq 0}^{i_1+i_2+\dots+i_{m-1}=k} f_{i_1} f_{i_2} \dots f_{i_{m-1}} f_{n-k} + \sum_{i_1,i_2,\dots,i_m, \geq 0}^{i_1+i_2+\dots+i_{m-1}=n} f_{i_1} f_{i_2} \dots f_{i_{m-1}} f_{n-n} +$$

$$\sum_{k=1}^{n} \sum_{i_1,i_2,\dots,i_m, \geq 0}^{i_1+i_2+\dots+i_{m-1}=k} f_{i_1} f_{i_2} \dots f_{i_{m-1}} f_{n-1-k} = S_m(n) + \sum_{k=1}^{n-2} \sum_{i_1,i_2,\dots,i_m, \geq 0}^{i_1+i_2+\dots+i_{m-1}=k} f_{i_1} f_{i_2} \dots f_{i_{m-1}} f_{n-1-k} +$$

$$\sum_{k=n-1}^{n} \sum_{i_1,i_2,\dots,i_m, \geq 0}^{i_1+i_2+\dots+i_{m-1}=k} f_{i_1} f_{i_2} \dots f_{i_{m-1}} f_{n-1-k} =$$

$$S_m(n)+S_m(n-1)+\sum_{i_1,i_2,\dots,i_m,\ge 0}^{i_1+i_2+\dots+i_{m-1}=n-1} f_{i_1}f_{i_2}\dots f_{i_{m-1}}f_{n-1-(n-1)}+\sum_{i_1,i_2,\dots,i_m,\ge 0}^{i_1+i_2+\dots+i_{m-1}=n} f_{i_1}f_{i_2}\dots f_{i_{m-1}}f_{n-1-n}=$$

$$S_m(n)+S_m(n-1)+\sum_{i_1,i_2,\dots,i_m,\ge 0}^{i_1+i_2+\dots+i_{m-1}=n} f_{i_1}f_{i_2}\dots f_{i_{m-1}}f_{-1}=S_m(n)+S_m(n-1)+$$

$$\sum_{i_1,i_2,\dots,i_m,\ge 0}^{i_1+i_2+\dots+i_{m-1}=n} f_{i_1}f_{i_2}\dots f_{i_{m-1}}=S_m(n)+S_m(n-1)+S_{m-1}(n).$$

Thus $S_m(n+1)=S_m(n)+S_m(n-1)+S_{m-1}(n),\, n,m\in\mathbb{N}.$ **(1)**

Using recursion **(1)** we will find consequtively explicit formulas for $S_2(n), S_3(n)$ and $S_4(n)$.

Note, that since $f_0=0$ then $h_n=S_3(n)=\sum_{i,j,k\ge 0}^{i+j+k=n} f_if_jf_k=\sum_{k=0}^{n}\sum_{i,j\ge 0}^{i+j=k} f_if_jf_{n-k}.$

Also we will use
short notation g_n for sum $S_2(n)$ and s_n for sum $S_4(n)$.
Namely, for $m=2,3,4$ **(1)** becomes, respectively,

$S_2(n+1)=S_2(n)+S_2(n-1)+S_1(n) \iff$
$g_{n+1}=g_n+g_{n-1}+f_n, n\in\mathbb{N},$ **(2)**

$S_3(n+1)=S_3(n)+S_3(n-1)+S_2(n) \iff$
$h_{n+1}=h_n+h_{n-1}+g_n,\ n\in\mathbb{N},$ **(3)**

$S_4(n+1)=S_4(n)+S_4(n-1)+S_3(n)$
$s_{n+1}=s_n+s_{n-1}+h_n,\ n\in\mathbb{N}.$ **(4)**

Consider now Fibonacci Operator F defined by $F(a_n):=a_{n+1}-a_n-a_{n-1}$ for any sequence $(a_n)_{n\ge 0}$ of real numbers and in particular for any integer k consider two special applications of operator F,namely:

1. $F(a_nf_{n+k})=a_{n+1}f_{n+1+k}-a_nf_{n+k}-a_{n-1}f_{n-1+k}=a_{n+1}(f_{n+1+k}-f_{n+k}-f_{n-1+k})+$
$a_{n+1}f_{n+k}+a_{n+1}f_{n-1+k}-a_nf_{n+k}-a_{n-1}f_{n-1+k}=(a_{n+1}-a_n)f_{n+k}+$
$(a_{n+1}-a_{n-1})f_{n-1+k}=$
$(a_{n+1}-a_n)f_{n+k}+(a_{n+1}-a_{n-1})(f_{n+1+k}-f_{n+k})=(a_{n+1}-a_{n-1})f_{n+k+1}-$
$(a_n-a_{n-1})f_{n+k}.$

So, $F(a_nf_{n+k})=(a_{n+1}-a_{n-1})f_{n+k+1}-(a_n-a_{n-1})f_{n+k}$ **(5)**

2. $F(a_nf_{n+k+1})=a_{n+1}f_{n+k+2}-a_nf_{n+k+1}-a_{n-1}f_{n+k}=a_{n+1}(f_{n+k+2}-f_{n+k+1}-f_{n+k})+$
$a_{n+1}f_{n+k+1}+a_{n+1}f_{n+k}-a_nf_{n+k+1}-a_{n-1}f_{n+k}=(a_{n+1}-a_n)f_{n+k+1}+$
$(a_{n+1}-a_{n-1})f_{n+k}.$

So, $F(a_nf_{n+k+1})=(a_{n+1}-a_n)f_{n+k+1}+(a_{n+1}-a_{n-1})f_{n+k}.$ **(6)**

Also note that $F(a_n)=0 \iff a_n=(a_1-a_0)f_n+a_0f_{n+1}($ can be proved by Math Induction)

$(c_2=a_0, c_1 \iff \begin{cases} c_2=a_0 \\ c_1+c_2=a_1 \end{cases}$

Now we ready to find $g_n,\ h_n$ and after s_n.

Applying **(5)** and **(6)** to $a_n = n$ we obtain**:**
$F(nf_{n+k}) = 2f_{n+k} - f_{n+k-1}$ and $F(nf_{n+k+1}) = f_{n+k+1} + 2f_{n+k}$ for any integer k.
Since for $k = 0$ we have $\boxed{F(nf_{n+1}) = f_{n+1} + 2f_n}$ and $\boxed{F(nf_n) = 2f_{n+1} - f_n}$ then
$F(nf_{n+1}) + 2F(nf_n) = f_{n+1} + 2f_n + 2(2f_{n+1} - f_n) = 5f_{n+1}$ and, therefore,

$$f_{n+1} = F\left(\frac{nf_{n+1} + 2nf_n}{5}\right). \qquad \textbf{(7)}$$

Also, $2F(nf_{n+1}) - F(nf_n) = 2f_{n+1} + 4f_n - (2f_{n+1} - f_n) = 5f_n \iff$

$$f_n = F\left(\frac{2nf_{n+1} - nf_n}{5}\right). \qquad \textbf{(8)}$$

Hence, **(2)** $\iff F(g_n) = F\left(\dfrac{2nf_{n+1} - nf_n}{5}\right) \iff$
$F\left(g_n - \dfrac{2nf_{n+1} - nf_n}{5}\right) = 0 \iff$
$g_n = \dfrac{2nf_{n+1} - nf_n}{5} + c_1 f_{n+1} + c_2 f_n$.
Since $g_0 = 0 = c_1 \cdot 1 + c_2 \cdot 0 \iff c_1 = 0$ and
$g_1 = 0 = c_1 \cdot 1 + c_2 \cdot 1 + \dfrac{2 \cdot 1 - 1}{5} \iff$
$0 = c_2 + \dfrac{1}{5} \iff c_2 = -\dfrac{1}{5}$ then $g_n = \dfrac{2nf_{n+1} - nf_n}{5} - \dfrac{f_n}{5} = \dfrac{2nf_{n+1} - (n+1) f_n}{5}$.

Thus, $$\boxed{g_n = S_2(n) = \frac{2nf_{n+1} - (n+1) f_n}{5}} \qquad \textbf{(9)}$$

and now we can find h_n.
Applying **(5)** and **(6)** to $a_n = n^2$ we obtain**:**
$F(n^2 f_{n+k}) = \left((n+1)^2 - (n-1)^2\right) f_{n+k+1} - \left(n^2 - (n-1)^2\right) f_{n+k} =$
$4nf_{n+k+1} - (2n-1) f_{n+k} \iff F(n^2 f_{n+k}) = 4nf_{n+k+1} - (2n-1) f_{n+k}$ **(10)**

and
$F(n^2 f_{n+k+1}) = \left((n+1)^2 - n^2\right) f_{n+k+1} + \left((n+1)^2 - (n-1)^2\right) f_{n+k} =$
$(2n+1) f_{n+k+1} + 4nf_{n+k}. \iff F(n^2 f_{n+k+1}) = (2n+1) f_{n+k+1} + 4nf_{n+k}$ **(11)**

In particular for $k = 0$ in **(10)** and **(11)** we obtain
$F(n^2 f_n) = 4nf_{n+1} - (2n-1) f_n$ and $F(n^2 f_{n+1}) = (2n+1) f_{n+1} + 4nf_n$.
Hence,
$F(2n^2 f_n) + F(n^2 f_{n+1}) = 8nf_{n+1} - (4n-2) f_n + (2n+1) f_{n+1} + 4nf_n = 10nf_{n+1} \iff$
$10nf_{n+1} = F(n^2 f_{n+1} + 2n^2 f_n) - 2f_n - f_{n+1} \iff$
$10nf_{n+1} = F(n^2 f_{n+1} + 2n^2 f_n) - 2F\left(\dfrac{2nf_{n+1} - nf_n}{5}\right) - F\left(\dfrac{nf_{n+1} + 2nf_n}{5}\right) \iff$
$10nf_{n+1} = F\left(n^2 f_{n+1} + 2n^2 f_n - \dfrac{2(2nf_{n+1} - nf_n)}{5} - \dfrac{nf_{n+1} + 2nf_n}{5}\right) =$

$$F\left(\left(n^2-n\right)f_{n+1}+2n^2f_n\right)\iff f_{n+1}=F\left(\frac{\left(n^2-n\right)f_{n+1}+2n^2f_n}{10}\right) \qquad \textbf{(12)}$$

and

$F\left(2n^2f_{n+1}\right)-F\left(n^2f_n\right)=(4n+2)f_{n+1}+8nf_n-\left(4nf_{n+1}-(2n-1)f_n\right)=$
$10nf_n-f_n+2f_{n+1}\iff F\left(2n^2f_{n+1}-n^2f_n\right)=10nf_n-f_n+2f_{n+1}\iff$
$10nf_n=F\left(2n^2f_{n+1}-n^2f_n\right)+f_n-2f_{n+1}\iff$
$10nf_n=F\left(2n^2f_{n+1}-n^2f_n\right)+F\left(\dfrac{2nf_{n+1}-nf_n}{5}\right)-F\left(\dfrac{2nf_{n+1}+4nf_n}{5}\right)=$
$F\left(2n^2f_{n+1}-n^2f_n+\dfrac{2nf_{n+1}-nf_n}{5}-\left(\dfrac{2nf_{n+1}+4nf_n}{5}\right)\right)=F\left(2n^2f_{n+1}-\left(n^2+n\right)f_n\right)\iff$

$$\boxed{nf_n=F\left(\frac{2n^2f_{n+1}-\left(n^2+n\right)f_n}{10}\right)=F\left(\frac{ng_n}{2}.\right).} \qquad \textbf{(13)}$$

Since $(n+1)f_n=nf_n+f_n=F\left(\dfrac{2n^2f_{n+1}-\left(n^2+n\right)f_n}{10}\right)+F\left(\dfrac{2nf_{n+1}-nf_n}{5}\right)=$
$F\left(\dfrac{2n^2f_{n+1}-\left(n^2+n\right)f_n}{10}+\dfrac{2nf_{n+1}-nf_n}{5}\right)=F\left(\dfrac{n\left(2(n+2)f_{n+1}-(n+3)f_n\right)}{10}\right)$
then using **(13)** , **(12)** and **(8)** we obtain $5g_n=2nf_{n+1}-(n+1)f_n=$
$F\left(\dfrac{\left(n^2-n\right)f_{n+1}+2n^2f_n}{5}\right)-F\left(\dfrac{2n^2f_{n+1}-\left(n^2+n\right)f_n}{10}\right)-F\left(\dfrac{2nf_{n+1}-nf_n}{5}\right)=$
$=F\left(\dfrac{\left(n^2-n\right)f_{n+1}+2n^2f_n}{5}-\dfrac{2n^2f_{n+1}-\left(n^2+n\right)f_n}{10}-\dfrac{2nf_{n+1}-nf_n}{5}\right)=$
$F\left(\dfrac{\left(5n^2+3n\right)f_n-6nf_{n+1}}{10}\right).$

That is $g_n=F\left(\dfrac{\left(5n^2+3n\right)f_n-6nf_{n+1}}{50}\right)$ and, therefore,

(2) $\iff F\left(h_n-\dfrac{\left(5n^2+3n\right)f_n-6nf_{n+1}}{50}\right)=0\implies$

$$h_n=\frac{\left(5n^2+3n\right)f_n-6nf_{n+1}}{50}+c_1f_{n+1}+c_2f_n.$$

Since $h_0=0=c_1$ and $h_1=0=\dfrac{2}{50}+c_2\iff c_2=-\dfrac{1}{25}$ then

$$h_n=\frac{\left(5n^2+3n\right)f_n-6nf_{n+1}}{50}-\frac{f_n}{25}=\frac{\left(5n^2+3n-2\right)f_n-6nf_{n+1}}{50}. \text{ Thus,}$$

$$\boxed{h_n=S_3(n)=\frac{\left(5n^2+3n-2\right)f_n-6nf_{n+1}}{50}}$$

Remark.
Now we consider another way to obtain g_n.
Note that $F(F(g_n))=F(f_n)=0$ and $F(F(F(h_n)))=F(F(g_n))=$
$F(f_n)=0.$

Since characteristic polynomials of $F(F(g_n))$ and $F(F(F(h_n)))$ are $(x^2-x-1)^2$ and $(x^2-x-1)^3$ respectively, then

$$g_n, h_n = P(n)\phi^n + Q(n)\overline{\phi}^n,$$

where $\phi, \overline{\phi}$ roots of equation $x^2-x-1-0$ and $P(x), Q(x)$ polynomials of degree that does not exceed 1 in the case of g_n and 2 in the case of h_n. Since ϕ^n and $\overline{\phi}^n$ can be represented as linear combination of f_{n+1} and f_n then also we can represent g_n, h_n in the form $P(n)\phi^n + Q(n)\overline{\phi}^n$, namely

$$g_n = (an+b) f_{n+1} + (cn+d) f_n = anf_{n+1} + (cn+d) f_n$$

(because $g_0 = 0$) and $h_n = (an^2+bn+c) f_{n+1} + (pn^2+qn+r) f_n = (an^2+bn) f_{n+1} + (pn^2+qn+r) f_n$ (because $h_0 = 0$),

where $g_0 = g_1 = 0, g_2 = 1, g_3 = 2, h_0 = h_1 = h_2 = 0, h_3 = 1, h_4 = 3$

Since $g_1 = 0 \iff a+c+d = 0,\ g_2 = 1 \iff 4a+2c+d = 1, g_3 = 2 \iff 9a+6c+2d = 2$ then $\begin{cases} 3a+c=1 \\ 7a+4c=2 \end{cases} \iff \begin{cases} a = 2/5 \\ c = -1/5 \end{cases} \implies d = -1/5$

and, therefore,

$$\boxed{g_n = \frac{2nf_{n+1} - (n+1) f_n}{5}.}$$

Or, we can obtain g_n *by substitution* $g_n = anf_{n+1} + (cn+d) f_n$ in $g_{n+1} - g_n - g_{n-1} = f_n$.

Indeed,

$a(n+1) f_{n+2} + (c(n+1)+d) f_{n+1} - anf_{n+1} - (cn+d) f_n - a(n-1) f_n - (c(n-1)+d) f_{n-1} = f_n \iff a(n+1)(f_{n+1}+f_n) + (c(n+1)+d) f_{n+1} - anf_{n+1} - (cn+d) f_n - a(n-1) f_n - (c(n-1)+d)(f_{n+1}-f_n) = f_n \iff (a+2c) f_{n+1} + (2a-c) f_n = f_n \implies$

$\begin{cases} a+2c=0 \\ 2a-c=1 \end{cases} \iff \begin{cases} a = 2/5 \\ c = -1/5 \end{cases}.$

Since F annulate the df_n then value of d we cańt obtain by this way. But we can use $g_1 = 0 \iff a+c+d = 0 \implies d = -1/5$.

Similarly, we can find h_n, namely, since F annulate the rf_n we can determine a, b, p, q by consideration identity

$F((an^2+bn) f_{n+1} + (pn^2+qn) f_n) = \dfrac{2nf_{n+1} - (n+1) f_n}{5}$ and after find r from condition $h_1 = 0 \iff a+b+p+q = 0$.

Consider now calculation of $s_n = S_4(n)$.

Applying **(5),(6)** for $a_n = n^3,\ k = 0$ we obtain

$$F(n^3 f_n) = \left((n+1)^3 - (n-1)^3\right) f_{n+1} - \left(n^3 - (n-1)^3\right) f_n = (6n^2+2) f_{n+1} - (3n^2-3n+1) f_n \quad \textbf{(14)}$$

$$F(n^3 f_{n+1}) = \left((n+1)^3 - n^3\right) f_{n+1} + \left((n+1)^3 - (n-1)^3\right) f_n = (3n^2+3n+1) f_{n+1} + (6n^2+2) f_n \quad \textbf{(15)}$$

Since $g_n = \dfrac{2nf_{n+1} - (n+1) f_n}{5},\ f_n = F(g_n), nf_n = F\left(\dfrac{ng_n}{2}\right)$ and $nf_{n+1} = F\left(\dfrac{(n^2-n) f_{n+1} + 2n^2 f_n}{10}\right)$ then

$F(2n^3 f_{n+1}) - F(n^3 f_n) = (6n^2+6n+2) f_{n+1} + (12n^2+4) f_n -$

$\left(\left(6n^2+2\right)f_{n+1}-\left(3n^2-3n+1\right)f_n\right)=15n^2f_n+6nf_{n+1}-(3n-5)f_n=$
$15n^2f_n+6F\left(\dfrac{\left(n^2-n\right)f_{n+1}+2n^2f_n}{10}\right)-3F\left(\dfrac{ng_n}{2}\right)+5F\left(g_n\right)\iff$
$15n^2f_n=F\left(2n^3f_{n+1}\right)-F\left(n^3f_n\right)-6F\left(\dfrac{\left(n^2-n\right)f_{n+1}+2n^2f_n}{10}\right)+$
$3F\left(\dfrac{ng_n}{2}\right)-5F\left(g_n\right)=$
$F\left(2n^3f_{n+1}-n^3f_n-\dfrac{3\left(\left(n^2-n\right)f_{n+1}+2n^2f_n\right)}{5}+\dfrac{(3n-10)}{2}\cdot\dfrac{2nf_{n+1}-(n+1)f_n}{5}\right)=$
$F\left(\dfrac{\left(10+7n-15n^2-10n^3\right)f_n+\left(20n^3-14n\right)f_{n+1}}{10}\right)$

Thus $15n^2f_n=F\left(\dfrac{\left(10+7n-15n^2-10n^3\right)f_n+\left(20n^3-14n\right)f_{n+1}}{10}\right)\iff$

$$n^2f_n=F\left(\frac{\left(10+7n-15n^2-10n^3\right)f_n+\left(20n^3-14n\right)f_{n+1}}{150}\right) \tag{16}$$

Since $S_3(n)=h_n=\dfrac{\left(5n^2+3n-2\right)f_n-6nf_{n+1}}{50}$, $nf_{n+1}=F\left(\dfrac{\left(n^2-n\right)f_{n+1}+2n^2f_n}{10}\right)$,
$nf_n=F\left(\dfrac{2n^2f_{n+1}-\left(n^2+n\right)f_n}{10}\right)$ then $F\left(S_4(n)\right)=S_3(n)\iff$
$F\left(s_n\right)=\dfrac{1}{10}\cdot n^2f_n+\dfrac{3}{50}\cdot nf_n-\dfrac{1}{25}f_n-\dfrac{3}{25}nf_{n+1}=$
$\dfrac{1}{10}\cdot F\left(\dfrac{\left(10+7n-15n^2-10n^3\right)f_n+\left(20n^3-14n\right)f_{n+1}}{150}\right)+\dfrac{3}{50}\cdot F\left(\dfrac{2n^2f_{n+1}-\left(n^2+n\right)f_n}{10}\right)-$
$\dfrac{1}{25}F\left(g_n\right)-\dfrac{3}{25}F\left(\dfrac{\left(n^2-n\right)f_{n+1}+2n^2f_n}{10}\right)=F(\dfrac{\left(10+7n-15n^2-10n^3\right)f_n+\left(20n^3-14n\right)f_{n+1}}{1500}+$
$\dfrac{3\left(2n^2f_{n+1}-\left(n^2+n\right)f_n\right)}{500}-\dfrac{g_n}{25}-\dfrac{3\left(\left(n^2-n\right)f_{n+1}+2n^2f_n\right)}{250})=$
$F(\dfrac{\left(10+7n-15n^2-10n^3\right)f_n+\left(20n^3-14n\right)f_{n+1}}{1500}+\dfrac{3\left(2n^2f_{n+1}-\left(n^2+n\right)f_n\right)}{500}-$
$\dfrac{2nf_{n+1}-(n+1)f_n}{125}-\dfrac{3\left(\left(n^2-n\right)f_{n+1}+2n^2f_n\right)}{250})=$
$F\left(\dfrac{\left(11+5n-30n^2-5n^3\right)f_n+\left(10n^3-10n\right)f_{n+1}}{750}\right)$

Hence, $s_n=\dfrac{\left(11+5n-30n^2-5n^3\right)f_n+\left(10n^3-10n\right)f_{n+1}}{750}+c_1f_{n+1}+c_2f_n$

Since $s_0=s_0=0$ then $c_1=0$ and $c_2=\dfrac{19}{750}$ and, therefore,
$s_n=\dfrac{\left(11+5n-30n^2-5n^3\right)f_n+\left(10n^3-10n\right)f_{n+1}}{750}+\dfrac{19f_n}{750}=$

$$\frac{\left(30+5n-30n^2-5n^3\right) f_n+\left(10n^3-10n\right) f_{n+1}}{750}=$$
$$\frac{(n-1)(n+1)\left(2nf_{n+1}-(n+6) f_n\right)}{150}.$$

Thus, $\boxed{s_n=S_4(n)=\frac{(n-1)(n+1)\left(2nf_{n+1}-(n+6) f_n\right)}{150}.}$ **(17)**

Remark.
Since $S_m(n)=P_m(n) f_{n+1}+Q_m(n) f_n$, where $P_m(x), Q_m(x)$ some polynomials of degree less then m (because $F_m(S_m(n))=0$, where $F_m=\underbrace{F\circ F\circ\ldots\circ F}_{m\ times}$ and charaterictic polynomial of F_m is $\left(x^2-x-1\right)^m$) then we can obtain all coefficients of $P_m(x), Q_m(x)$ (excluding free coefficients) by subsitution $S_m(n)$ in **(1)** (of course in supposition that we know $S_{m-1}(n)$ in the form $P_{m-1}(n) f_{n+1}+Q_{m-1}(n) f_n$, that is in supposition that we know polynomials $P_{m-1}(x), Q_{m-1}(x)$).
And after using $S_m(0)=S_m(1)=0$ we can detrmine free terms of both polynomials.
Since $P_m(n+1) f_{n+2}+Q_m(n+1) f_{n+1}-P_m(n) f_{n+1}-Q_m(n) f_n-$
$P_m(n-1) f_n-Q_m(n-1) f_{n-1}=P_m(n+1)\left(f_{n+1}+f_n\right)+Q_m(n+1) f_{n+1}-$
$P_m(n) f_{n+1}-Q_m(n) f_n-P_m(n-1) f_n-Q_m(n-1)\left(\left(f_{n+1}-f_n\right)\right)=$
$f_{n+1}\left(P_m(n+1)+Q_m(n+1)-P_m(n)-Q_m(n-1)\right)+$
$f_n\left(P_m(n+1)-Q_m(n)-P_m(n-1)+Q_m(n-1)\right)=$
$f_{n+1}\left(P_m(n+1)-P_m(n)+Q_m(n+1)-Q_m(n-1)\right)+$
$f_n\left(P_m(n+1)-P_m(n-1)-Q_m(n)+Q_m(n-1)\right)$
then $F(S_m(n))=P_{m-1}(n) f_{n+1}+Q_{m-1}(n) f_n$ implies

$$\begin{cases} P_m(n+1)-P_m(n)+Q_m(n+1)-Q_m(n-1)=P_{m-1}(n) \\ P_m(n+1)-P_m(n-1)-Q_m(n)+Q_m(n-1)=Q_{m-1}(n) \end{cases} \quad \textbf{(18)}$$

1. Mathematical Horizons, September,1966-Problem 55, Proposed by David M. Bloom.